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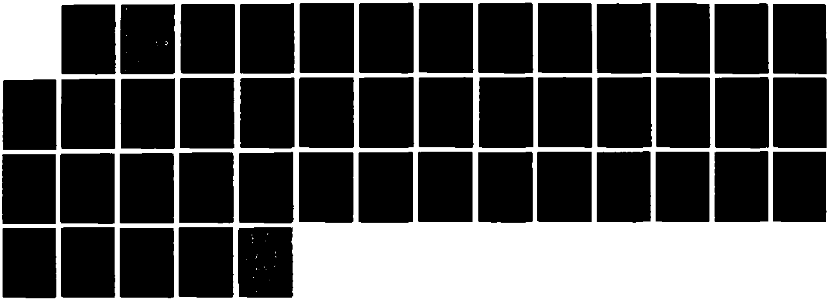

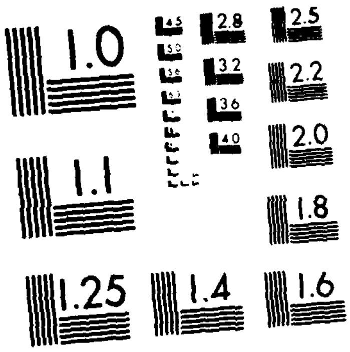

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Optimization of the Dilation Characteristic on the Control Jurisdiction of an Area Control Facility

Robert G. Mulholland

August 1987

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16. Abstract

In an air traffic control environment such as the National Airspace System the control function is based on stereographic representations of aircraft positions in a plane tangent to a sphere with a center collocated with the center of an ellipsoidal model of the geoid. The variation of the dilation (i.e., the discrepancy between the length of an infinitesimal arc on the model surface and the image of the arc in the plane) over the control jurisdiction of an air traffic control facility is one of many factors that adversely affect the ability of the facility to maintain separation standards. Techniques are disclosed for selecting a tangency point and a radius for the spherical support of the stereographic plane that minimize the variation of the dilation over the control jurisdiction about a predetermined constant. The constant can be viewed as a specification of the scale of the map that is the stereographic image in the plane of the portion of the surface of the earth model underlying controlled airspace.

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EXECUTIVE SUMMARY

In an air traffic control system where aircraft separation is maintained by controlling the trajectories of stereographic representations of aircraft in a plane tangent to a sphere with a center collocated with the center of the earth, dilation is one of many factors that adversely affect system performance. The length of the stereographic image of an arc on the mean sea level surface of the earth (i.e., the surface of an ellipsoid that approximates the geoid) differs from the length of the arc, and this dilation phenomenon is nonuniform in the sense that the lengths of the images of distinct equilength arcs need not be the same (i.e., the length of the image of an arc is a function of both the arc length and the position of the arc on the surface of the earth). As a result, the stereographic representation of the relative separation associated with a pair of aircraft at the same altitude varies with the absolute position of either member of the pair as well as with the relative separation itself. Also, the stereographic image of an aircraft exhibits an acceleration that reflects the change in the dilation with position rather than the physical characteristics of the actual flight trajectory. Indeed, if steps are not taken to minimize the variation of dilation over the portion of the surface of the earth underlying controlled airspace, dilation effects may seriously impact the design of automatic tracking and separation assurance features that are supposed to provide accurate forecasts of future positions of aircraft images in the stereographic plane.

The dilation at any point on the mean sea level surface of the earth is customarily expressed in terms of the ratio of the length of an infinitesimal arc through the point to the length of the stereographic image of the arc. The functional behavior of this measure of dilation over the control jurisdiction of a control facility is governed by the location of the tangency point and the radius of the sphere that supports the stereographic plane. The maximum value of the magnitude of the rate of change of the measure with respect to distance in the floor of the control jurisdiction (i.e., the portion of the surface of the earth underlying the control jurisdiction) is strongly dependent on the latitude and longitude of the tangency point. Since the magnitude of the acceleration of an aircraft image due to dilation tends to increase with the magnitude of the dilation rate, it makes sense to select a latitude-longitude pair for which the maximum value of the magnitude of the rate is small. Depending upon the shape of the floor, this criterion for tangency point selection may result in a latitude-longitude pair corresponding to a point in the floor or outside the floor. In any event, determination of a latitude-longitude pair for the tangency point automatically fixes the ratio of the maximum and minimum values of the dilation on the floor. Subsequent to the selection of the latitude and longitude of the point of tangency the absolute values of these extremums are completely controlled by the radius of the sphere that supports the stereographic

plane. Thus, the radius can be viewed as a parameter associated with the selection of a target value for dilation over the floor of the control jurisdiction.

This report discloses some techniques for selecting the tangency point and the spherical support radius for the stereographic plane. Among all possible latitude-longitude pairs that might be used to identify the tangency point there are some for which the maximum value of the magnitude of the dilation rate is least, and these are optimal in the sense that the corresponding acceleration induced in the stereographic plane is minimized. It is shown that there is only one optimal latitude-longitude pair in the case where the extent of the floor of the control jurisdiction falls within the design limit of the Advanced Automation System on the size of the coverage region of an area control facility. In addition, it is shown that the optimal latitude-longitude pair can be found as the solution to a linear programming problem whenever the extent of the floor is specified in terms of a finite number of points (e.g., the points on the mean sea level surface of the earth defined by the latitude-longitude pairs representing the locations of the radars that support the surveillance function of an area control facility or a finite collection of points that are more or less uniformly distributed over the floor). Since the solution of such a problem can be easily obtained by means of well known techniques (e.g., the simplex method) this result can be used to formulate an algorithm for determining the latitude and longitude of the tangency point. Also, under the assumption that the latitude and longitude of the tangency point are known, it is shown how to select the spherical support radius so that the maximum deviation of the dilation over the control jurisdiction from a predetermined design constant is a minimum. In effect, the spherical support radius is adjusted so that the constant falls midway between the minimum and maximum values of the dilation on the floor of the control jurisdiction, and it turns out that the magnitude of the ratio of the difference between the extremums to the constant is invariant to whatever numerical value is assigned to the constant. Finally, it is shown that this ratio is at most 0.0705 for any area control facility that might be commissioned in the Advanced Automation System. Consequently, the design constant can be viewed as an effective measure of the scale of the map formed by the image of the floor of the control jurisdiction in the stereographic plane.

1. INTRODUCTION.

In an air traffic control (ATC) environment like the National Airspace System (NAS) aircraft are separated in at least one of the three dimensions of altitude, latitude, and longitude by means of commands transmitted from a ground facility. Aircraft altitude is conveyed to the ground facility by voice communication or transponder replies to radar interrogation messages. The remaining dimensions are obtained indirectly through a projection algorithm that manipulates raw surveillance information (i.e., reported altitude and radar measurements of slant range and azimuth). The objective of the projection algorithm is to provide a point on a plane surface (i.e., the system plane of the control facility) that is a stereographic representation of aircraft latitude and longitude with respect to some ellipsoidal earth model. The control facility attempts to separate aircraft in one or both of the dimensions of latitude and longitude by keeping whatever points are provided by the projection scheme a prescribed distance (e.g., 5 nautical miles (nmi)) apart.

There are two factors, among many others, that adversely affect the ability of the control facility to maintain separation standards. These are projection error (i.e., the difference between the actual stereographic representation of an aircraft and the point representation provided by the projection algorithm) and dilation (i.e., the discrepancy between the distance separating points on the surface of the earth model and the distance between the stereographic representations of these points in the system plane). If the projection error is large then there is some question as to what is being controlled (i.e., latitude and longitude or something else). If it is small then there is no such question. Dilation adversely affects the ability of the control facility to maintain separation standards in the sense that it gives the facility a false impression of the actual distance measured over the surface of the earth model between one location and another that cannot be corrected by a constant scale factor.

Projection errors are pretty much a function of the projection algorithm. On the other hand, dilation is determined by the geometric relationship between the system plane and the earth model. In particular, the system plane is tangent to a conformal sphere (i.e., a sphere with a center collocated with that of the earth model) and dilation is dependent upon the location of the point of tangency and the radius of the sphere. In other words, there are two basic jobs involved in the design of a stereographic projection system in ATC applications. First, the designer must select a conformal sphere and a point of tangency that provide a reasonable dilation characteristic over the control jurisdiction. Second, the designer must provide an algorithm that is capable of converting raw surveillance information into a point on the system plane with little or no projection error. In the case of the Advanced Automation System

(AAS) it is a requirement that the projection error not exceed 0.005 nmi (reference 1) for any aircraft under radar control within the jurisdiction of an area control facility (ACF). There exists a projection algorithm that satisfies the AAS error requirement (reference 2). This report deals with some techniques for creating a near optimal dilation characteristic over the breadth of the floor of an ACF control jurisdiction (i.e., the portion of the surface of the earth model underlying the airspace controlled by the ACF).

The remainder of the report is divided into six sections. The next section is concerned with several aspects of the quantitative expression of dilation that can be exploited in the design of an ATC projection system. Although there exists an exact mathematical expression for the dilation at any point of an ellipsoidal earth model, it is not too useful in the design process. Section 3 deals with some practical formulas for estimating the dilation. In section 4 it is shown how these formulas can be employed in the selection of the radius of the sphere that supports the system plane under the assumption that the location of the point of tangency is known to within a radial line segment emanating from the center of the earth model. Section 5 is concerned with the problem of selecting this radial line segment, and there it is revealed how the selection process can be formulated as a linear programming problem. Regardless of what procedure is used to select the point of tangency and the radius of the spherical support for the system plane, the dilation will vary from point to point on the floor of the control jurisdiction. Also, the total variation (i.e., the difference between the maximum and minimum values of the dilation on the floor) tends to increase with increasing floor size. Section 6 discloses a constraint on the total variation implied by the AAS design limit with respect to the size of the surveillance coverage region associated with an ACF. Concluding remarks appear in section 7, and details of a mathematical nature are relegated to 10 appendices. Finally, numerical results presented in the report are based on an ellipsoidal earth model characterized by a polar radius of 3432.372 nmi and an equatorial radius of 3443.919 nmi (i.e., the earth model is assumed to be the reference ellipsoid chosen for the North American Datum of 1983 (reference 3)).

2. QUANTIFICATION OF DILATION.

The dilation m_p at a point p on the earth model can be quantified in terms of the length ds_1 of an infinitesimal arc on the model surface that contains the point and the length ds_2 of the stereographic representation of the arc in the system plane. The accepted measure of dilation is the limit of the ratio of ds_2 to ds_1 as the latter approaches zero. The limit can be expressed as a product of three factors (reference 4). One of these is the function

$$f(g_c) = 2/[1 + \cos(g_c)] \quad (1)$$

where g_c is the angle subtended at the center of the earth model by the point of tangency of the system plane and a conformal representation of the point p (i.e., the point on the surface of a conformal sphere at the conformal latitude and longitude of p). Another is the ratio of the radius E of the sphere that supports the system plane to the equatorial radius a of the earth model. The remaining factor is a function h(J) of the geodetic latitude J of p that increases monotonically from 1 to a number just under 1.0034 as the magnitude of the latitude increases from 0° to 90°. Unfortunately, this function is complicated, and as a result, the formula

$$m_p = h(J)(E/a)f(g_c) \quad (2)$$

is not too useful in the design of ATC projection systems.

Although the function h(J) is somewhat complicated, it is clear that dilation is nearly proportional to the simple function $f(g_c)$, and hence, the shape of the dilation characteristic over the floor of a control jurisdiction is strongly dependent on the parameter g_c. As will be seen, there are two aspects of this parameter that are significant from the standpoint of projection system design. One of these is based on the fact that there are infinitely many conformal representations of the same point on the surface of the earth model. The other stems from the fact that the angle subtended at the center of the earth model by two points on the model surface is essentially the same as the angle subtended at the same location by conformal representations of the points.

The angle g_c and the specific conformal sphere of radius E that supports the system plane are independent entities. There are as many conformal representations of a point on the surface of the earth model as there are conformal spheres, and all of these representations lie on the same radial line segment emanating from the center of the earth. Consequently, there are infinitely many conformal representations of the floor of a control jurisdiction, and regardless of whether the point p is a member of the floor, g_c is the unique angle subtended at the center of the earth model by the point of tangency and any conformal representation of p. Also, if q is the earth model representation of the point of tangency (i.e., the location on the model surface specified by the geodetic latitude and longitude of the tangency point) then the same angle is subtended at the center of the earth by any conformal representation of the point p and any conformal representation of q, including the tangency point itself. For example, the angle g_c is the same as the angle subtended at the center of the earth by the conformal representations of p and q on the unit sphere (i.e., the conformal sphere characterized by a radius of 1 in whatever units the dimension of length is expressed). This concept can be exploited in the selection of the tangency point.

The angle g_c is a rather abstract entity. Fortunately, it can be replaced with near impunity by other parameters of a more concrete nature. One of these is the angle g subtended at the center of the earth by the point p and the earth model representation q of the point of tangency. Another can be defined in terms of the great circle distance between two points on the surface of the earth model (i.e., the length of the shortest curve connecting the points that can be formed from the intersection of the model surface and the plane containing both points and the center of the earth). If the great circle distance d separating p from q is expressed in nmi then the ratio of d to 60 is a fairly accurate estimate of the degree measure of either of the angles g and g_c. As shown in appendix A, the magnitude of the difference between the ratio and g is less than 0.18 percent of the former. Likewise, if g is known then d is known to the extent that the magnitude of the difference between d and the product of 60 and g cannot exceed 0.18 percent of the product. The inequalities

$$\mathrm{mag}(g - g_c) \leq 0.000282 \tag{3}$$

and

$$\mathrm{mag}(d/60 - g_c) \leq 0.0018(d/60) + 0.000282 \tag{4}$$

express the accuracies of g and d/60 as estimates of g_c when distance is expressed in nmi and angle measures are specified in degrees. For example, if d is 1800 nmi then g_c is known to the extent that it is somewhere between 29.946° and 30.054°. On the other hand, if g is known to be 30° then g_c cannot differ from 30° by more than 0.000282°. A derivation of inequalities (3) and (4) is provided in appendix B.

The utility of the degree measure d/60 as an estimate of g_c does not depend so much on the difference between the measure and g_c as it does upon the difference between f(d/60) and f(g_c). Similarly, in the case of g as an estimate of g_c, the difference between f(g) and f(g_c) is of primary importance. Table 1 demonstrates that the replacement of g_c by d/60 or g in the dilation formula results in numbers that are essentially the same as m_p. In each row of the table the entry $B_1(d)$ in the third column is an upper bound on the magnitude of the difference between 1 and the ratio f(g_c)/f(d/60), and the entry in the fourth column upper bounds the magnitude of the difference between 1 and f(g_c)/f(g). In other words, the actual dilation at point p is not significantly different from the number generated by the dilation formula when g_c is replaced by either one of the estimates g and d/60. For example, if the point p is separated from the earth model representation of the point of tangency by a great circle distance of 1800 nmi then the magnitude of the difference between the actual dilation m_p at p and the estimate obtained from the dilation formula by replacing g_c with the ratio of 1800 to 60 is at most 0.025413 percent of the estimate. Appendix C discloses the procedure used to generate the bounds in

table 1. Approximations of angles like g_c by parameters of the type d/60 and g are useful in the selection of a near optimal radius for the conformal sphere that supports the system plane.

TABLE 1. ESTIMATION ACCURACIES OF d/60 AND g.

d (nmi)	g (deg)	$B_1(d)$ (10^{-7})	$B_2(g)$ (10^{-7})
60	1	3.2	0.4
120	2	11.9	0.9
180	3	26.0	1.3
240	4	45.6	1.7
300	5	70.8	2.2
600	10	279.4	4.3
900	15	627.5	6.5
1200	20	1117.7	8.7
1500	25	1753.9	10.9
1800	30	2541.3	13.2
2100	35	3486.4	15.5
2400	40	4597.3	17.9

3. ESTIMATION OF DILATION.

The dilation at any point in the floor of the control jurisdiction is closely approximated by the function

$$m(g_c) = k(E/a)f(g_c) \qquad (5)$$

where k is the arithmetic mean of upper and lower bounds in the interval from 1 to 1.0034 on the set of values of the function h(J) associated with the geodetic latitudes of the points in the floor. Indeed, if h_1 is a lower bound, h_2 is an upper bound, and

$$k = (h_1 + h_2)/2 \qquad (6)$$

then, as shown in appendix D, the dilation at any point in the floor must satisfy the inequality

$$\mathrm{mag}[m_p - m(g_c)] \leq [(h_2 - h_1)/(h_1 + h_2)]m(g_c). \qquad (7)$$

Thus, if the bounds are 1 and 1.0034 (i.e., k is 1.0017) then the use of $m(g_c)$ as an esimate of the dilation at any point on the surface of the earth model results in an estimation error that is at most 0.17 percent of the estimate. Table 2 implies that tighter bounds than these apply to smaller regions of the surface of the earth. Consequently, as demonstrated in examples 1 and 2, greater estimation accuracies can be realized over the floor of an ACF control jurisdiction.

TABLE 2. VARIATION OF h(J) WITH THE MAGNITUDE OF J.

mag(J) (deg)	h(J)
15	1.000223
25	1.000595
35	1.001097
45	1.001670
55	1.002245
65	1.002752
75	1.003130

Example 1.

If the floor falls between the north geodetic latitudes of 35° and 45° then the function h(J) is not less than 1.001097 nor greater than 1.001670 on the floor. Thus, the arithmetic mean of these numbers, 1.001384, is an acceptable value for the constant k, and the corresponding estimation error at any point on the floor cannot exceed 0.029 percent of $m(g_c)$.

Example 2.

If the floor extends from 5° south geodetic latitude to 15° north geodetic latitude then the minimum and maximum values of h(J) on the floor are 1 and 1.000223. Consequently, 1.000112 is the value to be assigned to k, and the maximum error associated with $m(g_c)$ as an estimate of the dilation at any point in the floor is 0.011 percent of the estimate.

From the point of view of a controller charged with the separation of aircraft it is important that the separation between points in the floor of the control jurisdiction be accurately portrayed by the separation of the stereographic representations of the points in the system plane. For this reason a constant dilation over the entire control jurisdiction is highly desirable. Unfortunately, the estimate $m(g_c)$, as well as the dilation itself, is a variable. In fact, the estimate increases from k(E/a) to infinity as the angle g_c increases from 0° to 180°. Thus, the magnitude of the difference between the dilation and a predetermined design constant n (i.e., a design goal for the value of dilation on the floor of the control jurisdiction) is as significant a design factor as the dilation itself. Consequently, if $m(g_c)$ is to serve as a useful estimate of dilation then

$$v(n,g_c) = \mathrm{mag}[m(g_c) - n] \tag{8}$$

must be close to

$$u_p(n) = \mathrm{mag}(m_p - n). \tag{9}$$

In appendix D it is shown that

$$\mathrm{mag}[u_p(n) - v(n,g_c)] \leq [(h_2 - h_1)/(h_1 + h_2)][v(n,g_c) + n] \quad (10)$$

for every point p in the floor of the control jurisdiction. As will be seen, it is possible to keep $v(n,g_c)$ below 0.0352n on the floor of an ACF control jurisdiction that fulfills the size requirements of the AAS specification. Stated another way, if the system plane point of tangency and the radius of the conformal sphere supporting the system plane are properly selected then $v(n,g_c)$ will differ from the magnitude of the difference between the dilation and n by at most

$$V(n) = 1.0352[(h_2 - h_1)/(h_1 + h_2)]n. \quad (11)$$

Since $h_2 - h_1$ is at most 0.0034 and $h_1 + h_2$ is never less than 2 it follows that V(n) is at most 0.176 percent of n. In practice, the value of V(n) can be made much smaller than this. For instance, V(n) is 0.000296n in example 1 and it is 0.000115n in example 2.

4. SELECTION OF THE SPHERICAL SUPPORT RADIUS.

Assuming that the earth model representation of the point of tangency is known (i.e., the angle g_c is defined) the estimate $m(g_c)$ can be viewed as a function of the parameters E and g_c, and it is clear that to each positive number that might be used as the radius of the spherical support for the system plane there corresponds some maximum value, other than 0, of $v(n,g_c)$ on the floor of the control jurisdiction. Fortunately, among all such numbers it is easy to locate one for which the maximum is least. As shown in appendix E, this optimal radius is

$$E_o(n,F_c,G_c) = 2n(a/k)/[f(F_c) + f(G_c)] \quad (12)$$

where F_c and G_c represent the minimum and maximum angles, respectively, subtended at the center of the earth model by the system plane point of tangency and a point within any conformal representation of the floor. Using r to represent the ratio of $f(G_c)$ to $f(F_c)$, the formula

$$v_o(n,r) = n[(r - 1)/(r + 1)] \quad (13)$$

supplies the corresponding maximum value of the magnitude of the difference between n and the estimate of dilation on the floor of the control jurisdiction. If the earth model representation of the point of tangency is located in the floor then $f(F_c)$ is 1 (i.e., F_c is 0) and $v_o(n,r)$ takes on the value

$$v_o(n,f(G_c)) = nw(G_c) \quad (14)$$

where

$$w(G_c) = [f(G_c) - 1]/[f(G_c) + 1]. \quad (15)$$

The inequality

$$0 \leq v_o(n,r) \leq nw(G_c) \qquad (16)$$

is a direct consequence of the fact that the function $f(g_c)$ increases as g_c increases from $0°$ to $180°$ and $v_o(n,r)$ is an increasing function of r. Thus, the dilation estimate associated with the optimal radius differs from the design constant n by at most $100w(G_c)$ percent of the constant at any point in the control jurisdiction.

While it is fairly obvious that the angle G_c is not too different from the largest angle G subtended at the center of the earth by a point in the floor of the control jurisdiction and the earth model representation q of the point of tangency, it is not altogether clear that the magnitude of the difference between the two angles is bounded above by the right side of the inequality (3) that expresses the accuracy of either one of the parameters g and g_c as an estimate of the other. The problem is that it may be difficult to establish the existance of a point t in the floor such that G is the angle subtended at the center of the earth by t and q and that G_c is the angle subtended at the same location by conformal representations of t and q. Similar remarks apply to the difference between F_c and the smallest angle F subtended at the center of the earth by q and a point in the floor. Nevertheless, it can be shown that the accuracy expressed by the inequality (3) for g as an estimate of g_c applies to F and G as estimates of F_c and G_c, respectively. Also, the accuracy expressed by the inequality (4) that is associated with d/60 as an estimate of g_c applies to the estimation of G_c by the product of 1/60 and the largest great circle distance D_G in nmi separating q from a point in the floor. Likewise, the inequality formed from (4) by replacing g_c with F_c and d with the shortest great circle distance D_F between q and a point in the floor expresses the accuracy of $D_F/60$ as an estimate of F_c. An argument supporting these assertions is provided in appendix F.

The utility of the degree measures $D_F/60$ and $D_G/60$ as estimates of the angles F_c and G_c does not depend so much on the difference between the measures and the angles as it does upon the dilations (or more precisely, the estimates of dilation) associated with the optimal radius $E_o(n,F_c,G_c)$ and the suboptimal radius $E_o(n,D_F/60,D_G/60)$. Similarly, the dilations associated with the optimal radius and the suboptimal radius $E_o(n,F,G)$ are of primary importance. Table 3 illustrates the difference between the estimates of the maximum deviation of the dilation from the design constant n on the floor associated with the optimal radius $E_o(n,0,G_c)$ and the suboptimal radii $E_o(n,0,D_G/60)$ and $E_o(n,0,G)$ in the case where the earth model representation of the tangency point is in the floor of the control jurisdiction. It also demonstrates the accuracy of $w(G)$ and $w(D_G/60)$ as estimates of $w(G_c)$. In each row of the table the entry $B_3(D_G)$ in the fourth column is an upper bound on the magnitude of the difference between $w(G_c)$ and $w(D_G/60)$, and the entry $B_4(G)$ in the fifth

column upper bounds the magnitude of the difference between $w(G_c)$ and $w(G)$. As demonstrated by the table, the estimate of the maximum deviation of the dilation from n associated with the optimal radius is not significantly different from the deviations associated with the suboptimal radii. In other words, if the great circle distances D_G and D_F are available then there is little reason for one to be reluctant about employing $E_o(n, D_F/60, D_G/60)$ as the radius of the spherical support for the system plane. Likewise, $E_c(n, F, G)$ can be used as the radius of the supporting sphere whenever the angles G and F are available. Appendix G discloses the procedure used to generate the bounds in table 3.

TABLE 3. BOUNDS ON mag[$w(G_c) - w(G)$] AND mag[$w(G_c) - w(D_G/60)$]

D_G (nmi)	G (deg)	$w(D_G/60)$ or $w(G)$ (10^{-4})	$B_3(D_G)$ (10^{-4})	$B_4(G)$ (10^{-4})
300	5.00	9.5	0.0354	0.0011
600	10.00	38.1	0.1397	0.0022
900	15.00	85.9	0.3137	0.0032
1200	20.00	153.1	0.5587	0.0043
1500	25.00	239.8	0.8764	0.0055
1800	30.00	346.5	1.2689	0.0066
1815	30.25	352.4	1.2906	0.0066
2100	35.00	473.5	1.7390	0.0077
2400	40.00	621.2	2.2892	0.0089

The parameters D_F, D_G, F, and G may not be readily available. For example, there is always some inherent distortion associated with any planar representation of a portion of the surface of the earth, and so it is not possible to obtain an exact determination of the distances D_F and D_G from a standard map illustrating the boundary of the floor of the control jurisdiction. On the other hand, the latitude and longitude of each radar in the network of radars supporting the surveillance function of an ACF will most certainly be known. This information together with the latitude and longitude of the tangency point can be used to compute the degree measures of the minimum and maximum angles A_{min} and A_{max} subtended at the center of the earth by the earth model representation of the point of tangency and a location on the model surface corresponding to the latitude and longitude of a member of the radar network. Also, it is most likely that some reasonable upper bound S on the ranges of the network radars will be available. For example, the maximum effective range of the search radars employed in NAS is often quoted as being 200 nmi. Consequently, the sum of the degree measure of A_{max} and the ratio of S in nmi to 60 can be viewed as a reasonable estimate of the degree measure of G. In like manner, the angle F can be taken to be the maximum of the numbers 0 and $A_{min} - S/60$. In those cases where the members of the radar network are not endowed with the same effective range, this technique can be extended to provide

even better estimates of F and G when the range of each radar is known. In fact, procedures of this nature may be the only means for acquiring realistic measures of F and G.

5. SELECTION OF THE TANGENCY POINT.

Since $v_o(n,r)$, the maximum deviation of the dilation estimate from the design constant n on the floor of the control jurisdiction that is associated with the optimal radius $E_o(n,F_c,G_c)$, is an increasing function of the ratio r of $f(G_c)$ to $f(F_c)$ and the parameter r is determined by the location of the tangency point, there is some reason to believe that from among all possible locations for the point of tangency one should be selected for which the ratio is least. Unfortunately, this may be a difficult problem to solve. Also, there is good reason to believe that it is not the proper problem to solve.

Both the dilation and the rate of change of dilation with respect to distance play important roles in projection system performance. As already indicated, departures of the dilation from the design constant n result in a stereographic image of air traffic that does not exactly portray the separation between aircraft relative to the earth model. On the other hand, departures of the dilation rate from 0 introduce undesirable acceleration effects in the system plane. The slope of the dilation estimate (i.e., the derivative of $m(g_c)$ with respect to the angle g_c) is a measure of dilation rate. Since

$$s(g_c) = m(g_c)\tan(g_c/2) \tag{17}$$

is the formula for the slope in $(\text{radians})^{-1}$, it is clear that large departures of the rate from 0 are possible. Example 3 illustrates the manner in which dilation rate affects system plane representations of moving vehicles. Indeed, if no effort is made to control the dilation rate on the floor of the control jurisdiction then it may be impossible to construct an automatic aircraft tracking system with current tracking technology that will satisfy the performance requirements of the AAS.

Example 3.

Suppose a particle is moving with constant speed over the surface of the earth model along the intersection of the model surface with a plane containing the earth model representation q of the point of tangency and the center of the earth. If the great circle distance d separating the particle from q is measured in nmi and c is the particle speed then (to the extent that d/60 represents the degree measure of the angle subtended at the center of the earth by the tangency point and a conformal representation of the particle) the corresponding speed of the system plane representation of the particle is nearly the same as the product of m(d/60) and c. In addition, the system plane representation of the particle is accelerating in the direction

of motion of the representation, and the magnitude of the acceleration is in the neighborhood of the product of 1/60, pi/180, s(d/60), and the square of c.

Minimization of the parameter r does not guarantee that the maximum value of the slope of the dilation estimate on the floor of the control jurisdiction will be a minimum. The maximum value of the estimate on the floor associated with the optimal radius is the sum of $v_o(n,r)$ and n. Thus,

$$s_o(n,r,G_c) = [v_o(n,r) + n]\tan(G_c/2) \qquad (18)$$

is the maximum value of the slope of the estimate on the floor associated with the optimal radius. Since G_c is not necessarily a monotone increasing function of r there is little reason to believe that the best position for the point of tangency is a location that renders r a minimum.

A good position for the tangency point is any location that minimizes the maximum angle G_c subtended by the point of tangency and a point in a conformal representation of the floor. Regardless of the location of the tangency point, the parameter r is at least 1 and at most $f(G_c)$. As a result,

$$s_o(n,f(G_c),G_c) = n[w(G_c) + 1]\tan(G_c/2) \qquad (19)$$

bounds the maximum slope $s_o(n,r,G_c)$ from above and

$$s_o(n,1,G_c) = n\tan(G_c/2) \qquad (20)$$

bounds it from below. Since $w(G_c)$ is an increasing function of G_c the same is also true for the magnitude of the difference between these bounds. Also, (19) implies that the maximum slope $s_o(n,r,G_c)$ differs from the lower bound by at most $100w(G_c)$ percent of that bound. Finally, as shown in the previous section, $nw(G_c)$ upper bounds $v_o(n,r)$. Thus, there is ample reason to view the function $w(G_c)$ as a measure of goodness of the location of the point of tangency, and much can be gained by selecting a location for the tangency point for which the angle G_c is least.

Table 3 illustrates that the sensitivity of the measure $w(G_c)$ to the position of the earth model representation of the point of tangency is not so critical that one needs to resort to powerful optimization techniques in order to find an appropriate location for the tangency point. For example, $w(D_m/60)$ increases from 0.02398 to 0.04735 as D_m increases from 1500 nmi to 2100 nmi. Thus, the rate of change of the measure per unit change in distance is roughly 0.00004 nmi^{-1} when the largest great circle distance separating a point in the floor of the control jurisdiction from the earth model representation of the point of tangency is in the neighborhood of 1800 nmi. Consequently, an 87 nmi movement of q will result in a change in the measure $w(D_m/60)$ of about 10 percent. In fact, the value of $w(D_m/60)$ increases

from 0.03465 to 0.03813 as D_Θ increases from 1800 nmi to 1887 nmi and it is 0.03135 when D_Θ is 1713 nmi. Table 4 provides a more detailed view of the sensitivity of $w(D_\Theta/60)$ to changes in the location of the earth model representation of the point of tangency in terms of the first derivative of the function

$$y(D_\Theta) = w([pi/180][D_\Theta/60]) \qquad (21)$$

and its natural logarithm . Except in those cases, if any exist, where there is an overriding need for strict optimality in terms of minimization of the angle G_c, it appears that a satisfactory

TABLE 4. DILATION SENSITIVITY

D_Θ (nmi)	$(d/dD_\Theta)y(D_\Theta)$ (10^{-6} nmi^{-1})	$(d/dD_\Theta)\log[y(D_\Theta)]$ (10^{-4} nmi^{-1})
300	6.4	66.7
600	12.7	33.4
900	19.1	22.3
1200	25.6	16.7
1500	32.2	13.4
1800	38.9	11.2
2100	45.8	9.7
2400	52.7	8.5

location for the tangency point can be found by means of the application of a reasonably healthy eyeball to a map (e.g., a Lambert conformal projection) of the floor of the control jurisdiction.

While there may be no overriding technical need for an explicit algorithm that automatically provides a tangency point that is in some sense optimal, a good algorithm is usually a welcome replacement for a subjective procedure that might go awry in the wrong hands. In the case of control jurisdictions meeting the size requirements of the AAS there exists an algorithm for determining an optimal tangency point under a rather practical set of conditions. As shown in appendix H, if the extent of the floor of the control jurisdiction is fairly represented by a topologically closed subset R of the surface of the earth model then there is one, and only one, location q_o for the earth model representation of the system plane point of tangency that is compatible with a design requirement that the maximum angle G_c subtended at the center of the earth by the tangency point and a point in any conformal representation of the floor be minimal. If the floor is closed (i.e., all boundary points of the floor are members of the floor) then R can be the floor itself. The set R can also be a finite set of points on the surface of the earth model (e.g., the set of all points on the model surface corresponding to the latitude-longitude pairs that define the locations of the radars that support the surveillance function of the control facility). With the exception of some special cases, it may be extremely difficult to compute the latitude and

longitude of q_o when the number of points in the set R is infinite. On the other hand, if R is finite then the determination of the latitude and longitude of q_o reduces to a linear programming problem. The following algorithm is a prescription for setting up the linear programming problem and acquiring the conformal latitude and longitude of q_o from the problem solution. A detailed development of the algorithm is provided in appendix I.

Algorithm. Tangency point selection via linear programming.

Input data for the algorithm consists of n pairs of numbers $(L_1, M_1), \ldots, (L_n, M_n)$ where L_k and M_k are the longitude and conformal latitude, respectively, of the kth $(k = 1, \ldots, n)$ member p_k of a finite set R of n points on the surface of the earth model that represents the extent of the floor of the control jurisdiction. In effect, the pair (L_k, M_k) represents the spherical coordinates of the conformal representation of p_k on the unit sphere. It is assumed that the conformal representation of the entire set R on the surface of the unit sphere is a subset of a hemisphere. Since the floor of a control jurisdiction meeting the AAS design limit on the size of the coverage region of an ACF is much less than half the surface of the earth model, this assumption is satisfied in any case where R is a subset of any portion of the model surface that qualifies as the floor of a control jurisdiction in the context of the AAS specification.

a. Construct a 3xn matrix B such that the elements of the kth column of the matrix are the Cartesian coordinates of the kth element of the set R on the unit sphere. Thus, the 3x1 matrix

$$\underline{b}_k = \begin{vmatrix} \cos(M_k)\cos(L_k) \\ \cos(M_k)\sin(L_k) \\ \sin(M_k) \end{vmatrix} \tag{22}$$

is the kth column of B.

b. Construct the nxn symmetric matrix A that is the result of premultiplying the matrix B by its transpose.

c. Letting $\underline{1}$ represent the nx1 matrix with identical elements equal to 1, $\underline{1}^T$ the transpose of $\underline{1}$, and $\underline{\emptyset}$ the nx1 matrix with identical elements equal to Ø, find an nx1 matrix $\underline{x}$ and a scalar r such that the scalar is a minimum subject to the constraints that r and all the elements of $\underline{x}$ are non-negative,

$$A\underline{x} - r\underline{1} = \underline{\emptyset}, \tag{23}$$

and

$$\underline{1}^T\underline{x} = 1. \tag{24}$$

This is a linear programming problem in n + 1 variables (i.e., r

and the n elements of the nx1 matrix $\underline{x}$) and it can be solved by means of the simplex method (reference 5).

d. Construct the 3x1 matrix

$$\underline{u} = (r^{-1/2})B\underline{x} \tag{25}$$

where $\underline{x}$ and r are solutions of the linear programming problem of step c. As shown in appendix I, $\underline{u}$ is a unit vector and the elements of $\underline{u}$ are the direction cosines of the optimal tangency point (i.e., the Cartesian coordinates of the optimal tangency point are just the elements of the 3x1 matrix formed by the scalar multiplication of $\underline{u}$ by the radius of the sphere that supports the system plane).

e. Letting u_j represent the element in the jth row of the 3x1 matrix $\underline{u}$, determine the longitude L and conformal latitude M of the earth model representation of the optimal tangency point from the relationships

$$M = \text{arc sin}(u_3), \tag{26}$$

$$u_1 = 0 \text{ and } u_2 = 0 \text{ implies } L = 0, \tag{27}$$

$$u_1 = 0 \text{ and } u_2 > 0 \text{ implies } L = 90^\circ, \tag{28}$$

$$u_1 = 0 \text{ and } u_2 < 0 \text{ implies } L = -90^\circ, \tag{29}$$

$$u_1 > 0 \text{ implies } L = \text{arc tan}(u_2/u_1), \tag{30}$$

and

$$u_1 < 0 \text{ implies } L = 180^\circ + \text{arc tan}(u_2/u_1). \tag{31}$$

6. DILATION AND THE AAS DESIGN LIMIT.

The AAS specification (reference 1, page 44) places a design limit of "2500X2500" square nmi on "the surveillance coverage area" of an ACF. Unfortunately, the description of the limit provided by the specification is not easy to interpret. There are many regions of the surface of the earth involving an area of 6,250,000 square nmi that cannot possibly serve as the floor of a control jurisdiction. Thus, it is unreasonable to view the limit as a restriction on area alone. In other words, there exists a factor of shape that cannot be ignored, and so it is reasonable to assume that the expression "2500X2500" refers to a square region with a side length of 2500 nmi. However, it is impossible to construct a square on the surface of an ellipsoidal earth model. As a result, the meaning of the design limit is somewhat unclear.

A reasonable interpretation of the design limit can be formulated in terms of the smaller of the two regions bounded by a simple

closed curve constructed on the surface of a specific conformal sphere. The radius of the sphere is the arithmetic mean of the equatorial and polar radii of the earth model (i.e., 3,438.146 nmi) and the curve consists of four great circle arcs each of which is exactly 2500 nmi in length. As shown in appendix J, the center of the smaller of the two regions bounded by the four arcs is separated from each vertex of the boundary by a great circle distance of 1,811.774 nmi. If the AAS design limit is interpreted to be the radial projection of the smaller region onto the surface of the earth model then the limit corresponds to the somewhat fuzzy, but nevertheless realistic, notion of a region of the model surface having a square-like perimeter that is approximately 10,000 nmi in length and a centrally located point that is roughly 1,812 nmi from each vertex of the perimeter. In fact, as shown in appendix J, adjacent vertices of the perimeter are separated by a great circle distance that is between 2,495.802 nmi and 2,504.198 nmi, and the great circle distance between the centrally located point and any vertex of the perimeter is not less than 1,808.732 nmi nor more than 1,814.817 nmi. Needless to say, the maximum great circle distance separating the centrally located point from any other point in the square-like region cannot exceed the upper bound of 1,814.817 nmi.

Some very definite implications with respect to dilation are associated with a design limit that is a region of the surface of the earth model with a centrally located point that is separated from any other point in the region by a great circle distance less than or equal to a prescribed maximum D_L that is itself known to be less than 1,815 nmi. Specifically, the floor of any ACF control jurisdiction commissioned in the AAS can be envisioned as being embedded in the design limit. Thus, regardless of the location of the point of tangency and the size of the radius of the sphere that supports the system plane, the euclidean norm of the difference between the dilation and the design constant n on the floor of the control jurisdiction will not exceed the maximum value of the norm of the difference on the design limit. Consequently, there is no reason why the norm of the difference on the floor should ever exceed the maximum value of the norm on the limit in the case where the point of tangency is centrally located with respect to the limiting region and the size of the conformal radius is the subobtimal value $E_o(n,0,D_L/60)$. Since D_L is less than 1815 nmi, it follows from table 3 that this maximum norm is less than 0.0352n. As a result, it can be concluded that the magnitude of the difference between the dilation and n on the floor of a control jurisdiction falling within the AAS design limit can be kept below 0.0352n by (1) assigning a location to the point of tangency such that among all possible positions for the tangency point the maximum angle subtended at the center of the earth by the assigned location and a point in a conformal representation of the floor is essentially minimal and (2) using either one of the numbers $E_o(n,D_F/60,D_G/60)$ and $E_o(n,F,G)$ as the radius of the conformal support for the system plane.

7. CONCLUDING REMARKS.

Selection of the location of the tangency point can be reduced to an optimization problem involving the conformal representation of the floor of the control jurisdiction on the surface of the unit sphere of euclidean 3-space. Each element of the sphere is a unit vector that can be viewed as the direction cosine vector of a possible location for the tangency point. The maximum angle subtended at the center of the sphere by any such element and a point in the conformal representation of the floor is a measure of the worth of that element as the direction cosine vector of the tangency point. Any unit vector for which the maximum angle is least has maximum worth in terms of limiting the magnitude of the rate of change of dilation with distance on the floor. In effect, minimizing the maximum angle is consistent with the notion that the best dilation characteristic over the control jurisdiction is a constant. In the case where the extent of the floor is represented by a closed subset of the surface of the earth model (i.e., all boundary points of the set are members of the set) and the size of the floor meets the design limit of the AAS specification there is one, and only one, unit vector for which the maximum angle is least. If the set is finite (e.g., the collection of points in the floor representing the radar sites that support the surveillance function of the control facility or any finite set of points that are distributed in a more or less uniform fashion over the floor) then this unique vector can be found in terms of the solution of a linear programming problem via the simplex method or any other technique that is available for solving such problems.

Regardless of what unit vector is used as the direction cosine vector of the tangency point, there is always some variation of the dilation over the floor of the control jurisdiction, and the difference between the maximum and minimum values of the dilation on the floor is proportional to the radius of the conformal sphere that supports the system plane. By means of a judicious choice of the conformal support radius it is possible to minimize the maximum deviation of the dilation on the floor from a prescribed design constant. This criterion for selection of the spherical support radius is consistent with the notion that the representation in the system plane of the separation associated with a pair of points in the floor should be close to a known multiple of the separation. For example, if the design constant is 1/185200 nmi per nmi then there should be some assurance that a distance of 5 centimeters between the system plane representations of two aircraft within the control jurisdiction is truely indicative of a separation of 5 nmi over the surface of the earth.

Finally, assuming that the target value for dilation over the floor of the control jurisdiction is n and that the floor meets the AAS design limit on the size of the coverage region of an ACF, it is possible to provide a tangency point and a spherical support radius such that the maximum dilation on the floor is at

most 1.0352n and the minimum dilation is at least 0.9648n. In other words, if the tangency point and the spherical support radius are selected in accord with the techniques outlined in this report then the dilation will deviate from the dilation design constant by at most 3.52 percent of the constant over the control jurisdiction of any ACF commissioned in the AAS.

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APPENDIX A. GREAT CIRCLE DISTANCE AND ANGLE

There exists a simple relationship between the great circle distance d in nmi between two points p and q on the surface of an ellipsoidal earth model characterized by equatorial and polar radii a and b, respectively, and the degree measure g of the angle subtended at the center of the earth model by the points. The distance d is the shorter of the two curves on the model surface connecting p and q that are formed by the intersection of the surface and the plane containing the points and the center of the earth. If g is known then d is also known to the extent that it must lie between b(pi/180)g and a(pi/180)g. Also, knowledge of d determines g to the extent that the latter is not greater than (d/b)(180/pi) nor less than (d/a)(180/pi). In other words, d and g satisfy the expressions

$$[b(pi/180)-60]g \leq d - 60g \leq [a(pi/180) - 60]g \qquad \text{(A-1)}$$

and

$$[(180/pi)/a - 1/60]d \leq g - d/60 \leq [(180/pi)/b - 1/60]d. \qquad \text{(A-2)}$$

In the case of the reference ellipsoid chosen for the North American Datum of 1983 (where a is 3443.919 nmi and b is 3432.372 nmi) these inequalities imply that 0.0018(60g) is an upper bound of mag(d - 60g) and that mag(g - d/60) is not greater than 0.0018(d/60).

APPENDIX B. ANGLE ESTIMATION ERROR

The angle g subtended at the center of the earth model by points p and q on the model surface is a fairly good estimate of the angle g_c subtended at the same location by conformal representations of the points. Since the longitude of a point on the earth model is the same as the longitude of the conformal representation of the point, the difference between the two angles is due solely to the fact that the geocentric and conformal latitudes of the same point on the model surface are not necessarily the same. In fact,

$$\cos(g) = \sin(x)\sin(y) + A[\cos(x)\cos(y)] \qquad \text{(B-1)}$$

where x and y are the geocentric latitudes of points p and q, respectively, and A is the cosine of the difference between the longitudes of the two points. The corresponding formula for $\cos(g_c)$ can be obtained from the expression for cos(g) by replacing x with the conformal latitude x_c of the point p and y with the conformal latitude y_c of the point q. Since the geocentric and conformal latitudes of a point on the surface of the earth differ by at most 0.000141° it is apparent that the difference between the angles g and g_c is small.

As already shown in appendix A, the magnitude of the difference between the degree measure of g and the product of 1/60 and the great circle distance d in nmi separating the points p and q is at most 0.18 percent of d/60, and so the ratio d/60 should provide a good estimate of g_c. In fact, since g_c - d/60 is the same as the sum of g_c - g and g - d/60 it follows that

$$\mathrm{mag}(g_c - d/60) \leq \mathrm{mag}(g_c - g) + \mathrm{mag}(g - d/60). \qquad \text{(B-2)}$$

As a result, $\mathrm{mag}(g_c - d/60)$ is bounded above by the sum of the bound 0.0018(d/60) on mag(g - d/60) and any upper bound on the degree measure of $\mathrm{mag}(g_c - g)$.

A numerical bound on the difference between the angles g and g_c can be derived in terms of the gradient vector of the function arc cos(g) with respect to the arguments x and y. Specifically, if C is an upper bound on the magnitude of the gradient vector for all possible values of x, y, and A then

$$\mathrm{mag}(g - g_c) \leq C[(x - x_c)^2 + (y - y_c)^2]^{1/2}. \qquad \text{(B-3)}$$

Since 0.000141° is an upper bound on the degree measures of $\mathrm{mag}(x - x_c)$ and $\mathrm{mag}(y - y_c)$ it follows that the degree measure of the magnitude of the difference between g and g_c cannot exceed the bound

$$B = C[2(0.000141)^2]^{1/2}. \qquad \text{(B-4)}$$

It only remains to provide a numerical value for C.

The square of the magnitude $M(A)$ of the gradient vector of arc cos(g) with respect to x and y is given by the expression

$$M^2(A) = N(A)/D(A) \qquad (B\text{-}5)$$

where $N(A)$ and $D(A)$ are the nonnegative functions defined by the formulas

$$N(A) = (1 + A^2)[\cos^2(x)\sin^2(y) + \sin^2(x)\cos^2(y)]$$

$$- 4A\sin(x)\cos(x)\sin(y)\cos(y) \qquad (B\text{-}6)$$

and

$$D(A) = 1 - [\sin(x)\sin(y) + A\cos(x)\cos(y)]^2. \qquad (B\text{-}7)$$

The parameter A is not greater than 1 nor less than -1, and so there is a number v between Ø and 1 such that

$$A = v(-1) + (1 - v)(1). \qquad (B\text{-}8)$$

Also, the values of the functions $N(A)/2$ and $D(A)$ at the extreme values of the argument A are

$$N(-1)/2 = D(-1) = \sin^2(x + y) \qquad (B\text{-}9)$$

and

$$N(1)/2 = D(1) = \sin^2(x - y). \qquad (B\text{-}10)$$

In addition, the second derivative of $N(A)/2$ with respect to A is nonnegative and the second derivative of $D(A)$ is nonpositive. In other words, $N(A)/2$ is a convex up function of A and $D(A)$ is a convex down function of A. It follows that

$$N(A)/2 \leq v[N(-1)/2] + (1 - v)[N(1)/2] \qquad (B\text{-}11)$$

and

$$D(A) \geq vD(-1) + (1 - v)D(1). \qquad (B\text{-}12)$$

Since the right side of each of these inequalities is equal to the sum of $(v)\sin^2(x + y)$ and $(1 - v)\sin^2(x - y)$ it is clear that the ratio of $N(A)/2$ to $D(A)$ is no greater than 1 for all A in the interval extending from -1 to 1. Consequently, the magnitude $M(A)$ of the gradient vector must be bounded above by the square root of 2. In other words, the number $2^{1/2}$ can be assigned to the bound C. The corresponding value of the bound B on mag($g - g_c$) is Ø.ØØØ282°. Thus, the degree measure of the magnitude of the difference between the angles g and g_c is bounded above by twice the limit of Ø.ØØØ141° that upper bounds the magnitude of the difference between the degree measures of the geocentric and conformal latitudes of any point on the surface of the earth model.

APPENDIX C. EFFECT OF ANGLE ESTIMATION ERROR ON THE COMPUTATION OF DILATION

If x is the degree measure of an angle that is less than 180°, y is the degree measure of a like angle, and x and y differ at most by a positive number p(x) such that the sum of x and p(x) is less than 180° then the ratio f(x) of 2 to 1 + cos(x) must satisfy the inequality

$$\text{mag}[f(y)/f(x) - 1] \leq B(x,p(x)) \tag{C-1}$$

where

$$B(x,p(x)) = f(x + p(x))/f(x) - 1. \tag{C-2}$$

Indeed, f(x) is a positive function of the degree measure x on the interval I of nonnegative numbers less than 180, and so

$$\text{mag}[f(y)/f(x) - 1] = [1/f(x)]\text{mag}[f(y) - f(x)]. \tag{C-3}$$

Also, the first and second derivatives of f(x) with respect to x are nonnegative functions on I. Thus,

$$\text{mag}[f(y) - f(x)] \leq f(x + p(x)) - f(x) \tag{C-4}$$

and the bound B(x,p(x)) follows directly upon dividing both sides of the inequality by f(x).

As shown in appendix B, the great circle distance d in nmi separating points p and q on the surface of the earth model and the degree measure of the angle g_c subtended at the center of the earth by conformal representations of p and q satisfy the inequality

$$\text{mag}[d/60 - g_c] \leq p_1(d/60) \tag{C-5}$$

where

$$p_1(d/60) = 0.0018(d/60) + 0.000282. \tag{C-6}$$

If the sum of d/60 and $p_1(d/60)$ is less than 180° (i.e., d is less than 10780.58 nmi) then

$$B_1(d) = B(d/60,p_1(d/60)) \tag{C-7}$$

is an upper bound of the magnitude of the difference between 1 and the ratio of $f(g_c)$ to f(d/60).

In appendix B it is shown that the degree measures of g_c and the angle g subtended at the center of the earth by the points p and q differ at most by

$$p_2(g) = 0.000282. \tag{C-8}$$

Consequently, if the degree measure of g is less than $180° - p_e(g)$ (i.e., 179.999718°) then

$$mag[f(g_c)/f(g) - 1] \leq B_e(g) \qquad (C-9)$$

where $B_e(g)$ is the same as $B(g, p_e(g))$.

APPENDIX D. ACCURACY OF THE DILATION ESTIMATE

Suppose that x_1 and x_2 are numbers such that the former is less than the latter, x_0 is halfway between the limits x_1 and x_2, t is somewhere between the limits, and n is any number inside or outside of the interval extending from x_1 to x_2. If both x_0 and t lie on the same side of n then the distance separating mag(x_0 - n) from mag(t - n) is the same as the distance separating x_0 from t. Otherwise, the distance separating mag(x_0 - n) from mag(t - n) cannot exceed half the distance separating x_1 from x_2. In any case, the distance separating x_0 and t can never be greater than half the distance between x_1 and x_2. In other words, the inequality

$$\mathrm{mag}[\mathrm{mag}(x_0 - n) - \mathrm{mag}(t - n)] \leq (x_2 - x_1)/2 \qquad \text{(D-1)}$$

is valid for any number n.

If factor h(J) in (2) is bounded below by h_1 and above by h_2 then the dilation m_p lies between the limits

$$x_1 = [h_1 m(g_c)]/k \qquad \text{(D-2)}$$

and

$$x_2 = [h_2 m(g_c)]/k. \qquad \text{(D-3)}$$

If, in addition, k is the arithmetic mean of h_1 and h_2 then $m(g_c)$ is halfway between the limits, and the distance separating mag(m_p - n) and mag[$m(g_c)$ - n] cannot exceed

$$(x_2 - x_1)/2 = (1/2)[(h_2 - h_1)/k]m(g_c). \qquad \text{(D-4)}$$

Moreover, if $m(g_c)$ is not less than n then it is the same as the sum of mag[$m(g_c)$ - n] and n. Otherwise, the sum exceeds $m(g_c)$. In any event,

$$(x_2 - x_1)/2 \leq (1/2)[(h_2 - h_1)/k](\mathrm{mag}[m(g_c) - n] + n) \qquad \text{(D-5)}$$

for any number n, including Ø.

APPENDIX E. OPTIMAL RADIUS

Suppose that $g_1(x)$ is an increasing function of x, $g_2(x)$ is a decreasing function of x, and the graphs of the two functions cross one another when the independent variable takes on the value x_o. Also, let $g(x)$ represent the maximum of the magnitude of the numbers $g_1(x)$ and $g_2(x)$. Clearly,

$$g(x) \geq g(x_o) \tag{E-1}$$

for all values of the argument x. Hence, the minimum value of $g(x)$ occurs at the intersection point x_o.

The minimum and maximum angles F_c and G_c, respectively, subtended at the center of the earth by the point of tangency and a point in a conformal representation of the floor of the control jurisdiction are completely determined by the location of the earth model representation q of the tangency point. Thus, once q has been determined, the maximum value of the magnitude of the difference between the dilation estimate and the design constant n on the floor can be viewed as a function $g(E)$ of the radius E of the sphere that supports the system plane. Since $f(g_c)$ defined by (1) increases as g_c moves from 0° to 180°, $g(E)$ is just the maximum of the magnitude of the numbers

$$g_1(E) = k(E/a)f(G_c) - n \tag{E-2}$$

and

$$g_2(E) = n - k(E/a)f(F_c). \tag{E-3}$$

Consequently, the value of the argument E that satisfies the equation

$$g_1(E) = g_2(E) \tag{E-4}$$

renders $g(E)$ a minimum.

APPENDIX F. ESTIMATION OF ANGULAR EXTREMUMS

Let ang(x,y) represent the degree measure of the angle subtended at the center of the earth by any two points x and y. Also, let G represent the least upper bound of the set A of all numbers of the form ang(q,p) where q is the earth model representation of the point of tangency and p is a point in the floor K of the control jurisdiction. If there is a point s in the floor such that ang(q,s) is the same as G then G is the maximum angle subtended at the center of the earth by q and a point in the floor. Otherwise, the notion of a maximum angle is meaningless and it is necessary to resort to the concept of a least upper bound. In any event, an inequality identical to (3) can be established for G and the least upper bound G_c of the set A_c of all numbers of the form $ang(q_c,p_c)$ where q_c is the tangency point and p_c is a member of the conformal representation K_c on the unit sphere of the floor K. Likewise, it can be shown that the same inequality applies to the greatest lower bounds F and F_c of the sets A and A_c, respectively.

The mapping ang(q,p) is a continuous function of p on the surface of the earth model, and the closure of K (i.e., the set of all points that are in the floor and the boundary of the floor) is a bounded closed set in euclidean 3-space. Thus, ang(q,p) attains absolute minimum and maximum values on the closure of the floor, and so there exists a point t in the closure such that ang(q,t) is G. The inequality (3) implies that

$$ang(q_c,t_c) \geq G - z \qquad \text{(F-1)}$$

where t_c is the conformal representation on the unit sphere of t and the symbol z represents 0.000282°. Since the relationship between geodetic and conformal latitudes is a homeomorphism (i.e., a bijective function for which both the function and its inverse are continuous) the same is true of the transformation T that maps the surface of the earth model into the conformal representation of the model surface on the unit sphere. Thus, the closure of the conformal representation K_c of the floor is the image of the closure of the floor under the mapping T. Consequently, t_c is a member of the closure of K_c, and it follows that

$$ang(q_c,t_c) \leq G_c. \qquad \text{(F-2)}$$

Clearly, (F-1) contradicts (F-2) under the assumption that G exceeds the sum of G_c and z. As a result, it must be concluded that

$$G \leq G_c + z. \qquad \text{(F-3)}$$

The inequality

$$F \geq F_c - z \qquad \text{(F-4)}$$

can be established by a similar argument. It remains to show that G is not less than G_c - z and that F is not greater than F_c + z.

By virtue of the continuity of the function ang(q_c,p_c) on the unit sphere, there exists at least one point r_c in the closure of K_c such that ang(q_c,r_c) is the same as G_c. Also, the fact that the closure of K_c is the image of the closure of K under the transformation T implies that the point r_c is the image under T of some point r in the closure of the floor. Thus, by virtue of inequality (3),

$$\mathrm{ang}(q_c, r_c) \leq G + z. \qquad \text{(F-5)}$$

Under the assumption that G is less than G_c - z the inequality (F-5) implies that ang(q_c,r_c) is strictly less than G_c. Since this result contradicts the fact that ang(q_c,r_c) is exactly equal to G_c, it follows that

$$G \geq G_c - z. \qquad \text{(F-6)}$$

The inequality

$$F \leq F_c + z \qquad \text{(F-7)}$$

can be established in a similar fashion.

Let dist(x,y) represent the great circle distance in nmi between points x and y on the surface of the earth model. Also, let D_G and D_F be the least upper bound and greatest lower bound of the set of all numbers of the type dist(q,p) where, as before, q is the earth model representation of the tangency point and p is a member of the floor. Clearly,

$$\mathrm{mag}(D_G/60 - G_c) \leq \mathrm{mag}(D_G/60 - G) + \mathrm{mag}(G - G_c), \qquad \text{(F-8)}$$

$$\mathrm{mag}(D_F/60 - F_c) \leq \mathrm{mag}(D_F/60 - F) + \mathrm{mag}(F - F_c), \qquad \text{(F-9)}$$

and each of the factors mag(G - G_c) and mag(F - F_c) is bounded above by z. Hence, in order to establish an inequality like (4) for the pair D_G and G_c and the pair D_F and F_c, it is only necessary to show that mag(D_G/60 - G) is bounded above by 0.18 percent of D_G/60 and that 0.18 percent of D_F/60 is an upper bound of mag(D_F/60 - F).

As already pointed out, there is a member t of the closure of the floor such that ang(q,t) is G. Also, dist(q,p) is a continuous function of p on the surface of the earth model, and so there is a point s in the closure of the floor such that dist(q,s) is D_G. Thus,

$$b(\mathrm{pi}/180)\mathrm{ang}(q,s) \leq D_G \leq a(\mathrm{pi}/180)\mathrm{ang}(q,s) \qquad \text{(F-10)}$$

and

$$b(pi/180)G \le dist(q,t) \le a(pi/180)G \tag{F-11}$$

where a is the equatorial radius of the earth model in nmi and b is the polar radius in nmi. Since D_G is at least dist(q,t) and ang(q,s) cannot exceed G, it follows that

$$b(pi/180)G \le D_G \le a(pi/180)G. \tag{F-12}$$

Following the argument in appendix A, it is now a simple matter to show that

$$mag(D_G/60 - G) \le 0.0018(D_G/60) \tag{F-13}$$

and

$$mag(60G - D_G) \le 0.0018(60G). \tag{F-14}$$

In like manner, it can be shown that $mag(D_F/60 - F)$ cannot exceed 0.18 percent of $D_F/60$ and $mag(60F - D_F)$ is never greater than 0.18 percent of 60F.

APPENDIX G. EFFECT OF ANGULAR EXTREMUM APPROXIMATIONS ON THE ESTIMATION OF DILATION EXTREMUMS

If x and y represent the degree measures of two angles, the sum of x and an upper bound p(x) on the magnitude of the difference between x and y is no greater than 124.16°, and w(x) is the ratio of f(x) − 1 to f(x) + 1 where f(x) represents the function 2/[1 + cos(x)] then

$$\mathrm{mag}[w(x) - w(y)] \leq C(x,p(x)) \qquad \text{(G-1)}$$

where

$$C(x,p(x)) = w(x + p(x)) - w(x). \qquad \text{(G-2)}$$

The bound C(x,p(x)) is a result of the fact that both the first and second derivatives of w(x) with respect to x are nonnegative functions on the interval extending from 0° to 124.16°.

Let D_G represent the least upper bound in nmi of the set of great circle distances separating the earth model representation of the point of tangency from a point in the floor of the control jurisdiction. Also, let G_c represent the least upper bound of the set of degree measures of the angles subtended at the center of the earth by the tangency point and a point in any conformal representation of the floor. As shown in appendix F,

$$\mathrm{mag}(G_c - D_G/60) \leq p_1(D_G/60) \qquad \text{(G-3)}$$

where

$$p_1(D_G/60) = 0.0018(D_G/60) + 0.000282. \qquad \text{(G-4)}$$

Consequently, if the sum of $D_G/60$ and $p_1(D_G/60)$ is not greater than 124.16° (i.e., $D_G \leq 7{,}436.20$ nmi) then

$$B_3(D_G) = C(D_G/60,\ p_1(D_G/60)) \qquad \text{(G-5)}$$

is an upper bound of the magnitude of the difference between $w(G_c)$ and $w(D_G/60)$.

Let G represent the least upper bound of the set of degree measures of the angles subtended at the center of the earth by the earth model representation of the tangency point and a point in the floor. As shown in appendix F,

$$\mathrm{mag}(G_c - G) \leq p_2(G) \qquad \text{(G-6)}$$

where

$$p_2(G) = 0.000282. \qquad \text{(G-7)}$$

Thus, the magnitude of the difference between $w(G_c)$ and $w(G)$ is bounded above by

$$B_4(G) = C(G, p_2(G)) \tag{G-8}$$

whenever the sum of G and $p_2(G)$ does not exceed 124.16°.

APPENDIX H. UNIQUENESS OF THE OPTIMAL TANGENCY POINT

The results of this appendix are based on two assumptions with respect to the topology of the subset R of the surface of the earth model that is used to describe the extent of the floor of the control jurisdiction. First, the earth model is viewed as an ellpsoid with center collocated with the origin of euclidean 3-space E^3, and it is assumed that R is closed with respect to the topology induced in E^3 by the euclidean norm. Second, it is assumed that the size of the set R is restricted in the sense that there exists a vector $\underline{z}$ (i.e., a member of E^3 with a length of 1) in the conformal reoresentation R_c of R on the unit sphere and a positive number t, not greater than 1, such that the inner product $(\underline{z},\underline{y})$ of $\underline{z}$ and any vector $\underline{y}$ in R_c is not less than t. In other words, the maximum angle subtended at the origin (i.e., the center of the earth) by the point representation of $\underline{z}$ and any point in the conformal representation R_c of the floor is less than 90°. Since the size of a coverage region of an ACF meeting the design limit of the AAS is a good deal less than half the surface of the earth model, the existance of the unit vector $\underline{z}$ and the positive number t is assured for any subset of the surface of the earth that fairly represents the extent of the floor of a control jurisdiction that might eventually be commissioned in the AAS.

Under the assumptions of the preceding paragraph this appendix supports two assertions concerning the tangency point of the system plane. First, it is shown that there exists a unit vector $\underline{x}_o$ such that the smallest inner product in the set $\{(\underline{x}_o,\underline{y}):\underline{y}$ in $R_c\}$ is at least as great as the smallest inner product in the set $\{(\underline{x},\underline{y}):\underline{y}$ in $R_c\}$ for any unit vector $\underline{x}$. Since the inner product of unit vectors $\underline{x}$ and $\underline{y}$ is just the cosine of the angle subtended at the center of the earth by the point representations of the two vectors on the unit sphere, it follows that $\underline{x}_o$ is a conformal representation of the earth model representation q_o of an optimal tangency point. In particular, the earth model representation of $\underline{x}_o$ is just the image of $\underline{x}_o$ under the inverse of the bijection T that maps the surface of the earth model into the conformal representation of the model surface on the unit sphere. In effect, the geocentric latitude of $\underline{x}_o$ is the conformal latitude of $T^{-1}(\underline{x}_o)$ (i.e., q_o) and the longitude of $\underline{x}_o$ is the longitude of $T^{-1}(\underline{x}_o)$. Thus, the determination of $\underline{x}_o$ is equivalent to the determination of an optimal tangency point. Second, it is shown that $\underline{x}_o$ is a scalar multiple of the unique point $\underline{w}$ in the convex hull of the conformal representation of R that is nearest the origin of E^3. This result implies that there is one, and only one, point on the surface of the earth model for which the maximum angle subtended at the center of the earth model by any conformal representation of the point and a member of any conformal representation of the set R is least, and this point is q_o. In other words, there is only one point on the surface of the conformal sphere supporting the system plane that meets the criteria of an optimal tangency point, and this point is the product of $\underline{w}$ and the ratio of the radius of the spherical support

to the norm of $\underline{w}$.

The conformal representation R_c of the set R is closed and bounded. Specifically, the norm of each element of R_c is bounded above by 1, and so R_c is certainly bounded. Also, the bijection T and its inverse T^{-1} are continuous functions. Consequently, since R is closed, the image of R under the transformation T (i.e., R_c) is itself closed.

Among all the vectors in the convex hull $H(R_c)$ of the set R_c there exists one, and only one, vector $\underline{w}$ with a norm that is less than the norm of any other vector in the hull. The assumption that there is a unit vector $\underline{z}$ and a positive number t such that

$$\underline{y} \text{ in } R_c \text{ implies } (\underline{z},\underline{y}) \geq t \tag{H-1}$$

guarantees that R_c, and hence, the convex hull of R_c, is a subset of the convex set

$$J = \{\underline{y}\text{: norm}(\underline{y}) \leq 1 \text{ and } (\underline{z},\underline{y}) \geq t\}. \tag{H-2}$$

Since t exceeds Ø, the origin of E^3 is not a member of J. Also, since R_c is closed and bounded, the set $H(R_c)$ is closed and bounded. Thus, by virtue of a well known theorm on the minimum distance to a convex set (Luenberger, D., Optimization by Vector Space Methods, John Wiley, 1969, page 69) there exists one, and only one, vector $\underline{w}$ in the convex hull of R_c such that

$$\underline{y} \text{ in } R_c \text{ implies } (\underline{w},\underline{y}) \geq [\text{norm}(\underline{w})]^2, \tag{H-3}$$

and the norm of $\underline{w}$ is strictly less than the norm of any other member of $H(R_c)$.

If $\underline{x}$ is a unit vector then there exists a vector $\underline{s}(\underline{x})$ in R_c such that

$$\underline{y} \text{ in } R_c \text{ implies } (\underline{x},\underline{s}(\underline{x})) \leq (\underline{x},\underline{y}), \tag{H-4}$$

and $(\underline{x},\underline{s}(\underline{x}))$ is a continuous function of $\underline{x}$ on the set U of all unit vectors. As a result, there exists a unit vector $\underline{x}_o$ and a member $\underline{y}_o$ of R_c such that

$$(\underline{x}_o,\underline{y}) \geq (\underline{x}_o,\underline{y}_o) \geq \min\{(\underline{x},\underline{y})\text{: } \underline{y} \text{ in } R_c\} \tag{H-5}$$

for every unit vector $\underline{x}$ and every member $\underline{y}$ of R_c. The existance of $\underline{s}(\underline{x})$ is guaranteed by the fact the inner product $(\underline{x},\underline{y})$ is a continuous function of $\underline{y}$ and the set R_c is both closed and bounded (i.e., a continuous function attains an absolute minimum and an absolute maximum on a closed and bounded set in E^n). Also, like R_c, the set U is closed and bounded. Thus, the product set UxR_c is a closed and bounded subset of the product space E^3xE^3. Since the inner product of two vectors is a continuous function on this space, the inner product is uniformly continuous on UxR_c. As a result, the function $(\underline{x},\underline{s}(\underline{x}))$ is

continuous on the set U, and so it must attain an absolute maximum at some point $\underline{x}_o$ of U. Finally, the vector y_o is just the vector $s(x_o)$.

The inner product of $\underline{x}_o$ and $\underline{y}_o$ is equal to the norm of the minimum norm vector $\underline{w}$ in the convex hull of R_c. The inequality (H-3) implies that

$$([\text{norm}(\underline{w})]^{-1}\underline{w},\underline{y}) \geq \text{norm}(\underline{w}) \qquad \text{(H-6)}$$

for all $\underline{y}$ in R_c. Since $[\text{norm}(\underline{w})]^{-1}\underline{w}$ is a unit vector, it follows directly from (H-5) that

$$(\underline{x}_o,\underline{y}_o) \geq \text{norm}(\underline{w}). \qquad \text{(H-7)}$$

In order to establish that $(\underline{x}_o,\underline{y}_o)$ cannot exceed norm($\underline{w}$) it is sufficient to recognize that (H-5) implies that R_c, and hence, the convex hull of R_c, is a subset of the convex set

$$K = \{\underline{y}\text{: norm}(\underline{y}) \leq 1 \text{ and } (\underline{x}_o,\underline{y}) \geq (\underline{x}_o,\underline{y}_o)\}. \qquad \text{(H-8)}$$

Clearly,

$$\text{norm}^2(\underline{y} - (\underline{x}_o,\underline{y}_o)\underline{x}_o) = \text{norm}^2(\underline{y}) + (\underline{x}_o,\underline{y}_o)^2 - 2(\underline{x}_o,\underline{y}_o)(\underline{x}_o,\underline{y})$$

$$\leq \text{norm}^2(\underline{y}) - (\underline{x}_o,\underline{y}_o)^2 \qquad \text{(H-9)}$$

for any member $\underline{y}$ of K. Hence, among all elements of K the norm of the vector $(\underline{x}_o,\underline{y}_o)\underline{x}_o$ is least. The inequality

$$\text{norm}(\underline{w}) \geq (\underline{x}_o,\underline{y}_o) \qquad \text{(H-10)}$$

follows directly from the fact that the convex hull of R_c is a subset of K and $\underline{w}$ is a member of $H(R_c)$.

The vectors $\underline{w}$ and $[\text{norm}(\underline{w})]\underline{x}_o$ are the same. Specifically, the square of the norm of the difference between these two vectors can be expressed in the form

$$\text{norm}^2([\text{norm}(\underline{w})]\underline{x}_o - \underline{w})$$

$$= 2[\text{norm}(\underline{w})][\text{norm}(\underline{w}) - (\underline{x}_o,\underline{w})]. \qquad \text{(H-11)}$$

Since $\underline{w}$ is in the convex hull of R_c and the latter is a subset of K, it follows from (H-8) that $(\underline{x}_o,\underline{w})$ is at least $(\underline{x}_o,\underline{y}_o)$ which is itself the same as norm($\underline{w}$). Thus, the norm of the difference between $\underline{w}$ and $[\text{norm}(\underline{w})]\underline{x}_o$ is 0.

APPENDIX I. A LINEAR PROGRAMMING APPROACH TO TANGENCY POINT SELECTION

In this appendix it is assumed that the extent of the floor of the control jurisdiction is fairly represented by a finite set R of n points on the surface of the earth model. For example, each member of R might be a location on the model surface defined by the geodetic latitude and longitude of a radar or it might consist of a group of points more or less uniformly scattered over the floor. In addition, it is assumed that there exists at least one unit vector $\underline{z}$ such that the angle subtended at the center of the earth by $\underline{z}$ and any one of the n members of the conformal representation R_c of R on the unit sphere is less than $90°$. As pointed out in appendix H, this assumption is satisfied in any case where R fairly represents the extent of the floor of a control jurisdiction meeting the AAS design limit on the size of the coverage region of an ACF. Also, as established in appendix H, the closed nature of R and the existance of the vector $\underline{z}$ imply that the optimal tangency point is merely a scalar multiple of the minimum norm vector $\underline{w}$ in the convex hull $H(R_c)$ of R_c. Consequently, the longitude of $\underline{w}$ is the longitude of the earth model representation q_o of the optimal tangency point, and the geocentric latitude of $\underline{w}$ is the conformal latitude of q_o. As will be seen, this appendix discloses a practical technique for finding $\underline{w}$, and hence, q_o, in terms of the solution of a linear programming problem.

The set R_c is represented by a 3xn matrix B. The kth column of B is a function of the longitude L_k and the conformal latitude M_k of the kth member of the set R that describes the extent of the floor of the control jurisdiction. As indicated in figure I-1, the center of the earth is viewed as the origin of a Cartesian coordinate system with axes 1,2, and 3 where axis 3 coincides with the polar axis of the earth and the positive direction along axis 3 is from the South Pole to the North Pole. The element b_{jk} in the jth row and kth column of the matrix B is defined in terms of the formulas

$$b_{1k} = \cos(M_k)\cos(L_k), \qquad (I-1)$$

$$b_{2k} = \cos(M_k)\sin(L_k), \qquad (I-2)$$

and

$$b_{3k} = \sin(M_k). \qquad (I-3)$$

Thus, the kth column of the matrix B is a 3x1 matrix $\underline{b}_k$ with elements b_{1k}, b_{2k}, and b_{3k}.

The mimimum norm member of the convex hull $H(R_c)$ of R_c is a 3x1 matrix obtained by postmultiplying B by an nx1 probability matrix. An nx1 matrix $\underline{x}$ is a probability matrix if all components of the matrix are non-negative (denoted by $\underline{x} \geq \emptyset$) and a sum of 1 is obtained when the transpose $\underline{x}^T$ of $\underline{x}$ is

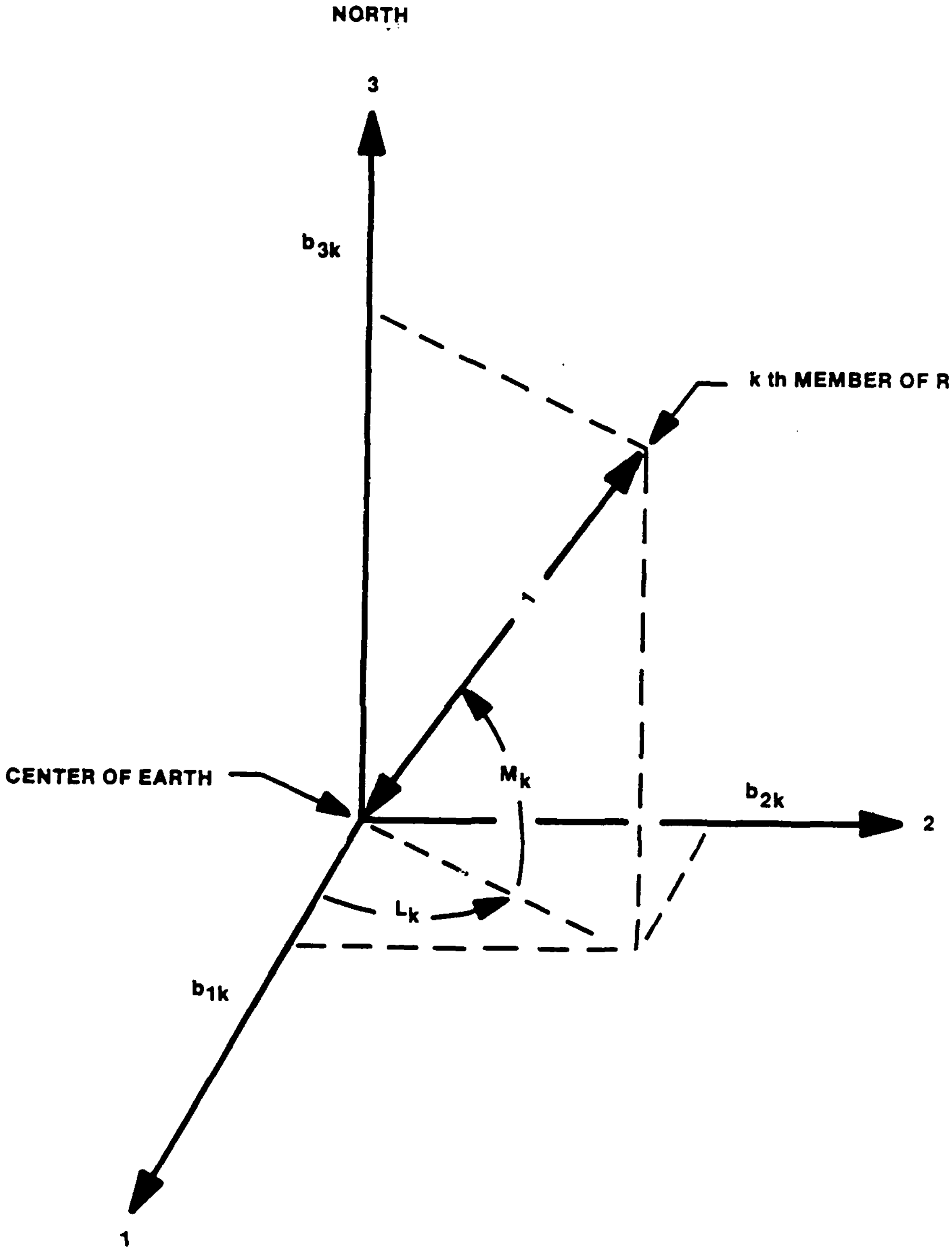

FIGURE I-1. COORDINATES OF kTH MEMBER OF THE SET R_c

postmultiplied by the nx1 vector $\underline{1}$ with identical components equal to 1. Since the convex hull of a finite set is just the collection of all convex linear combinations of the elements of the set, $H(R_c)$ is equivalent to the set of all 3x1 matrices of the form $B\underline{x}$ such that $\underline{x} \geq \emptyset$ and $\underline{x}^T\underline{1} = 1$.

Letting A represent the nxn symmetric semipositive definite matrix formed by premultiplying B by its transpose B^T, the minimum norm element of the convex hull of R_c is a 3x1 matrix of the form $B\underline{x}$ where $\underline{x}$ is any nx1 matrix that minimizes the function

$$G(\underline{x}) = \underline{x}^T A \underline{x} \tag{I-4}$$

subject to the constraints

$$\underline{x}^T\underline{1} - 1 = \emptyset \tag{I-5}$$

and

$$\underline{x} \geq \emptyset. \tag{I-6}$$

The constraints merely restrict $\underline{x}$ to the set of nx1 probablity matrices. The function $G(\underline{x})$ is the result of premultiplying the 3x1 matrix $B\underline{x}$ by its transpose, and so it is just the square of the norm of $B\underline{x}$.

Every solution of the constrained minimization problem is a probability matrix $\underline{x}$ satisfying the equation

$$A\underline{x} - r\underline{1} = \underline{\emptyset} \tag{I-7}$$

where r is a scalar and $\underline{\emptyset}$ is the nx1 null matrix (i.e., the elements of $\underline{\emptyset}$ are identical and equal to $\emptyset$). This result follows directly from a straight forward application of the generalized method of Lagrange multipliers to the problem of minimizing $G(\underline{x})$ subject to the constraints (I-5) and (I-6).

Any solution of the linear programing problem in n + 1 variables of minimizing the linear functional r subject to the constraints

$$\left|\begin{array}{c|c} A & \underline{1} \\ \hline \underline{1}^T & \emptyset \end{array}\right| \left|\begin{array}{c} \underline{x} \\ \hline -r \end{array}\right| = \left|\begin{array}{c} \underline{\emptyset} \\ \hline 1 \end{array}\right| \tag{I-8}$$

and

$$\left|\begin{array}{c} \underline{x} \\ \hline r \end{array}\right| \geq \emptyset. \tag{I-9}$$

is a solution to the problem of minimizing $G(\underline{x})$ subject to the constraints (I-5) and (I-6). The scalar r in equation (I-7) is merely the square of the norm of $B\underline{x}$. This fact can be verified

by premultiplying (I-7) by the transpose of the probability vector $\underline{x}$. In other words, the desired solution of (I-7) is the (n + 1)x1 matrix formed from the concatenation of the components of the vector $\underline{x}$ and the scalar r in which the scalar is minimal subject to the constraints that the vector is a probability matrix (i.e., the components of $\underline{x}$ are non-negative and sum to 1) and the scalar, the square of a norm, is non-negative. These 2 constraints and equation (I-7) are embodied in the matrix relationships (I-8) and (I-9). A solution to the linear programming problem is guaranteed by the fact that it involves the minimization of a functional that is bounded below by Ø.

APPENDIX J. SPHERICAL SQUARES

If x is a positive number not greater than 90 then there exists a spherical square on the surface of a conformal sphere in the sense that it is possible to select four distinct points p_0, p_1, p_2, and p_3 (i.e., the vertices of the square) on the surface of the sphere such that x is the degree measure of the angle subtended at the center of the earth by any pair of adjacent vertices (i.e., vertices $p_{m(i)}$ and $p_{m(i+1)}$ where i is any nonnegative integer and $m(i)$ represents the integer i modulo 4). The spherical coordinates of a point on the sphere consist of the radius of the sphere, the latitude of the point, and the longitude of the point. Let $x/2$ represent the latitude of p_1 and p_2, $-(x/2)$ the latitude of p_0 and p_3, L the longitude of p_2 and p_3, and $-L$ the longitude of p_0 and p_1. The degree measure A of the angle subtended at the center of the earth by p_1 and p_2 is the same as the degree measure of the angle subtended at the same location by p_0 and p_3. The cosine of this angle is just the inner product of the direction cosine vectors of p_1 and p_2. Thus,

$$\cos(A) = \cos^2(x/2)[2\cos^2(L) - 1] + \sin^2(x/2), \qquad \text{(J-1)}$$

and so $\cos(A)$ decreases from 1 to $-\cos(x)$ as L increases from $0°$ to $90°$. Since x is not greater than $90°$ it follows that A increases from $0°$ to at least $90°$ as L moves from $0°$ to $90°$. Consequently, there exists a degree measure L_0 not greater than $90°$ such that A is x when L takes on the value L_0, and (J-1) implies that

$$\cos^2(L_0) = \cos(x)/\cos^2(x/2). \qquad \text{(J-2)}$$

In other words, the points p_0, p_1, p_2, and p_3 become the vertices of a spherical square when L is L_0.

There is a point t on the surface of the sphere associated with the spherical square defined by the vertices p_0, p_1, p_2, and p_3 that is centrally located with respect to the square in the sense that the degree measure B of the angle subtended at the center of the earth by t and a vertex of the square is the same for all vertices of the square, the cosine of B is the square root of the cosine of the angle subtended at the center of the earth by adjacent vertices, and the angle subtended at the center of the earth by t and any point p on the shortest great circle arc connecting adjacent vertices is at most B. Let t be the point on the sphere corresponding to $0°$ latitude and $0°$ longitude. From considerations of symmetry it is apparent that the degree measure B of the angle subtended at the center of the earth by t and a vertex is the same for all vertices. Also, the cosine of B is just the inner product of the direction cosine vectors of t and any one of the vertices, and so

$$\cos(B) = \cos(x/2)\cos(L_0) = [\cos(x)]^{1/2}. \qquad \text{(J-3)}$$

Finally, let p be any point on the sphere with longitude L_o and a latitude of y degrees such that the magnitude of y does not exceed half the degree measure x of a side of the square. If C is the angle subtended at the center of the earth by t and p then

$$\cos(C) = \cos(y)\cos(L_o). \qquad (J-4)$$

Since mag(y) is not greater than x/2 it follows that C cannot exceed B.

Suppose now that the vertices of the spherical square are located on the surface of a conformal sphere with a radius R that is the arithmetic mean of the equatorial radius a and the polar radius b of the earth model, q_i is the radial projection of the vertex p_i onto the model surface, and s is the radial projection of the center t of the square onto the surface of the earth model. Clearly, the angle subtended at the center of the earth by any pair of vertices is the same as the angle subtended at the same location by the projections of the vertices onto the earth model. Likewise, the angle subtended at the center of the earth by t and a vertex of the square is the same as the corresponding angle formed by s and the projection of the vertex onto the model surface. Consequently, if D is the great circle distance between the adjacent vertices $p_{m(i)}$ and $p_{m(i+1)}$ of the spherical square, then the degree measure of the angle subtended at the center of the earth by these points is

$$x = (D/R)(180/pi), \qquad (J-5)$$

and the great circle distance separating the points $q_{m(i)}$ and $q_{m(i+1)}$ on the surface of the earth model is not greater than a(pi/180)x nor less than b(pi/180)x.

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